Multifractality and nonextensivity at the edge of chaos of unimodal maps

E. Mayoral and A. Robledo

Instituto de Física,
Universidad Nacional Autónoma de México,
Apartado Postal 20-364, México 01000 D.F., México.
E-mail addresses: estela@eros.pquim.unam.mx, robledo@fisica.unam.mx

Abstract

We examine both the dynamical and the multifractal properties at the chaos threshold of logistic maps with general nonlinearity \( z > 1 \). First we determine analytically the sensitivity to initial conditions \( \xi_t \). Then we consider a renormalization group (RG) operation on the partition function \( Z \) of the multifractal attractor that eliminates one half of the multifractal points each time it is applied. Invariance of \( Z \) fixes a length-scale transformation factor \( 2^{-\eta} \) in terms of the generalized dimensions \( D_{\beta} \). There exists a gap \( \Delta \eta \) in the values of \( \eta \) equal to \( \lambda_q = 1/(1 - q) = D_{\infty}^{-1} - D_{-\infty}^{-1} \) where \( \lambda_q \) is the \( q \)-generalized Lyapunov exponent and \( q \) is the nonextensive entropic index. We provide an interpretation for this relationship - previously derived by Lyra and Tsallis - between dynamical and geometrical properties.

Key words: Edge of chaos, multifractal attractor, nonextensivity

PACS: 05.10.Cc, 05.45.Ac, 05.90.+m

1 Introduction

The thermodynamic framework for the characterization of multifractals [1] has proved to be a useful and insightful tool for the analysis of many complex system problems in which this geometrical feature plays a prominent role in establishing physical behavior. This has clearly been the case of dynamical systems, turbulence, growth models, sand pile models, etc. [2]. As it has recently been pointed out, multifractality is also of relevance to critical [3] and glassy dynamics [4]. In the former case the geometry of critical fluctuations is found to be multifractal and this has a hold on their dynamical properties [3], whereas in the latter case slow dynamics develops as phase-space accessibility is reduced and is finally confined to a multifractal subspace [4]. Concurrently,
in the study of the same types of systems the nonextensive generalization of
the Boltzmann Gibbs (BG) statistical mechanics [5], [6] has recently raised
much interest and provoked considerable debate [7] as to whether there is
firm evidence for its applicability in circumstances where a system is out of
the range of validity of the canonical BG theory. Slow dynamics is plausi-
bly related to hindrance of movement in phase space that leads to ergodicity
breakdown, and this in turn has been linked to the failure of BG statistics [4].
Here we briefly examine the relationship between multifractal properties and
nonextensive dynamical evolution at the edge of chaos, a nonergodic state, in
unimodal maps, as part of a wider analysis [8], that might contribute to reveal
the physical source of the nonextensive statistics.

2 Nonextensivity of the Feigenbaum attractor

An important class of multifractals are the strange attractors that are generated
by iteration of nonlinear maps. Amongst these, a prototypical example is the
Feigenbaum attractor, also known as the edge of chaos, generated by the
logistic map,
\[ f_{\mu,2}(x) = 1 - \mu x^2, \quad -1 \leq x \leq 1, \]
when the control parameter \( \mu \) takes the value \( \mu_\infty = 1.40115\ldots \) [1]. Recently, the predictions of the nonexten-
sive theory have been rigorously proved [9] [10] for this state that marks the
transition from periodic to chaotic motion and that is characterized by the
vanishing of the ordinary Lyapunov exponent \( \lambda_1 = 0 \). At the edge of chaos
(and also at infinitely many other critical states of the same map such as the
pitchfork and tangent bifurcations where \( \lambda_1 = 0 \) [11]) the sensitivity to initial
conditions \( \xi_t \equiv |dx_t/dx_0| \) (where \( x_t \) is the orbit position at time \( t \) given the
initial position \( x_0 \) at time \( t = 0 \)) generally acquires power law instead of ex-
ponential time dependence. The nonextensive formalism suggests [12] that \( \xi_t \) when \( \lambda_1 = 0 \) obeys the \( q \)-exponential expression,
\[
\xi_t = \exp_q(\lambda_q t) \equiv [1 - (q - 1)\lambda_q t]^{-1/(q-1)},
\]
containing the entropic index \( q \) and the \( q \)-generalized Lyapunov coefficient \( \lambda_q \).
The BG exponential form \( \xi_t = \exp(\lambda_1 t) \) is recovered when \( q \to 1 \). Further, it
was conjectured [12] that the ordinary Pesin identity \( \lambda_1 = K_1 \), where \( K_1 \) is the
Kolmogorov-Sinai (KS) rate of entropy increase [13] is replaced by \( \lambda_q = K_q \),
where \( K_q t = S_q(t) - S_q(0), t \) large, and where
\[
S_q \equiv \sum_i p_i \ln_q^{(1/p_i)} = \frac{1 - \sum_i W_i^q}{q - 1}
\]
is the Tsallis entropy (\( \ln_q y \equiv (y^{1-q} - 1)/(1-q) \) is the inverse of \( \exp_q(y) \)).
In the limit \( q \to 1 \) \( K_q \) becomes \( K_1 \equiv t^{-1}[S_1(t) - S_1(0)] \) where \( S_1(t) = \frac{1}{1 - \sum_i W_i^1} \).
\[ \sum_{i=1}^{W} p_i(t) \ln p_i(t) \] In Eq. (2) \( p_i(t) \) is the probability distribution obtained from the relative frequencies with which the positions of an ensemble of trajectories occur within cells \( i = 1, \ldots, W \) at iteration time \( t \). Based on the renormalization group (RG) transformation \( R \equiv f(x) \equiv \alpha f(f(x/\alpha)) \) where \( \alpha \approx 2.5029 \) is the Feigenbaum’s universal constant that measures the power-law period-doubling spreading of iterate positions, a rigorous analytical confirmation of Eq. (1) and also of \( \lambda_q = K_q \) have been obtained \cite{9} \cite{10}. Specifically, \( \lambda_q \) and \( q \) are simply given by 
\[ \lambda_q = \ln \alpha / \ln 2 \approx 1.3236 \] and 
\[ q = 1 - \ln 2 / \ln \alpha \approx 0.2445. \]

On the other hand, a remarkable link between the well-known \cite{1} geometrical description of multifractals and the nonextensive statistics has been derived and corroborated numerically not only for the Feigenbaum attractor but for the edge of chaos of the generalization of the logistic map to nonlinearity \( z > 1 \), \( f_{\mu,z}(x) = 1 - \mu |x|^z, -1 \leq x \leq 1 \) \cite{14}. This consists of the expression
\[ \frac{1}{1-q} = \frac{1}{D_\infty} - \frac{1}{D_{-\infty}}, \tag{3} \]
where \( D_\infty \) and \( D_{-\infty} \) are the extreme values of the generalized dimension spectrum \( D_\beta \), and correspond to the smallest and largest length scales of the multifractal set \cite{1}. For the \( z \)-logistic map one has \( D_\infty = \ln 2 / z \ln \alpha \) and \( D_{-\infty} = \ln 2 / \ln \alpha \) (notice that \( \alpha \) depends on \( z \)) \cite{1}. In view of the clear association of the nonextensive statistics with the properties at the edge of chaos of the logistic map we have pursued the study of this critical state by means of the standard thermodynamic tools that characterize multifractals. Here we advance some of the results we have found \cite{8}.

## 3 Sensitivity to initial conditions

We determine the index \( q \) and the \( q \)-Lyapunov coefficient \( \lambda_q \) at the edge of chaos of the \( z \)-logistic maps through direct evaluation of the sensitivity to initial conditions \( \xi_t \). By making use of the properties \cite{15} \( g(0) = (-\alpha)^n g^{(2^n)}(0) = 1 \), \( g(1) = (-\alpha)^n g^{(2^n)}(\alpha^{-n}) = -\alpha^{-1} \) and
\[ g'(1) = (-1)^n g'(\alpha^{-n}) g(g(\alpha^{-n})) \cdots g(g^{(2^n-1)}(\alpha^{-n})) = -\alpha^{-z-1}, \tag{4} \]
of the fixed-point map, \( g(x) = \lim_{n \to \infty} (-\alpha)^n f_{\mu,z}^{(2^n)}(-x/\alpha^n) \), into the definition of \( \xi_t \),
\[ \xi_t \equiv \left| \frac{dx_t}{dx_0} \right| = \left| \frac{dg^{(2^n-1)}(x)}{dx} \right|_{x=x_0}, \tag{5} \]
Fig. 1. a) Absolute values of $|x_t|$ vs $t$ in logarithmic scales for the orbit with initial condition $x_0 = 0$ at $\mu_\infty$ of the logistic map $z = 2$. The labels indicate iteration time $t$. b) Same as a) with $|x_t|$ replaced by $1 - |x_t|$.

with the choice $x_0 = 0$, and for iteration times of the form $t = 2^n - 1$, we obtain $\xi_t = \alpha^{(z-1)n}$. This result can be re-expressed as

$$\xi_t = \exp_q(\lambda_q t) \equiv [1 - (q-1)\lambda_q t]^{-1/(q-1)}, \quad t = 2^n - 1, \quad (6)$$

with

$$\lambda_q = \frac{1}{1 - q} = (z - 1)\frac{\ln \alpha}{\ln 2}. \quad (7)$$

For details see Ref. [8]. Thus, without reference to the multifractal properties of the attractor, we have corroborated the result in Eq. (3) [14] and obtained additionally the $q$-Lyapunov coefficient $\lambda_q = (1 - q)^{-1}$. Alternatively, $\xi_t$ can be obtained from $|\Delta x_t|/|\Delta x_0|$ in the limit $\Delta x_0 \to 0$ where $\Delta x_t$ is the distance between two orbits at time $t$ when they were a distance $\Delta x_0$ at $t = 0$. Following the procedure described in Ref. [9] with the choices $|\Delta x_0| = \alpha^{-j} - \alpha^{-i}$, $i, j > 0$ and $t = 2^n - 1$, $n = 0, 1, \ldots$, we obtain $|\Delta x_t| = |\Delta x_0| \alpha^{(z-1)n}$ [8]. Next we study a multifractal property of the attractor and make contact again with Eq. (3).

4 Scaling properties at the edge of chaos

In Fig. 1a we have plotted (in logarithmic scales) the first few (absolute values) of iterated positions $|x_t|$ of the orbit at $\mu_\infty$ starting at $x_0 = 0$ where the labels indicate iteration time $t$. Notice the structure of horizontal bands, and that, in the top band lie half of the attractor positions (odd times), in the second band a quarter of the attractor positions, and so on. The top band is eliminated by
functional composition of the original map, that is by considering the orbit generated by the map \( g^{(2)}(0) \) instead of \( g(0) \). Successive bands are eliminated by considering the orbits of \( g^{(2k)}(0) \), \( k = 1, 2, \ldots \). The top band positions (odd times) can be reproduced approximately by the positions of the band below it (times of the form \( t = 2 + 4n, n = 0, 1, 2\ldots \)) by multiplication of a factor equal to \( \alpha \), e.g. \( |x_1| \simeq \alpha |x_2| \). Likewise, the positions of the second band are reproduced by the positions of the third band under multiplication by \( \alpha \), e.g. \( |x_2| \simeq \alpha |x_4| \), and so on. In Fig. 1b we show a logarithmic scale plot of \( 1 - |x_t| \) that displays a band structure similar to that in Fig. 1a, in the top band lie again half of the attractor positions (even times) and below the other half (odd times) is distributed in the subsequent bands. This time the positions in one band are reproduced approximately from the positions of the band lying below it by multiplication of a factor \( \alpha^z \), e.g. \( 1 - |x_3| \simeq \alpha^z (1 - |x_5|) \). The scaling properties amongst iterate positions merely follow from repeated composition of the map (for more details see Ref. [8]) and suggest the construction of an RG transformation described below. For this purpose it is convenient to write the two scale factors \( \alpha \) and \( \alpha^z \) as \( 2^{\eta_{-\infty}} \) and \( 2^{\eta_{\infty}} \), respectively, where \( \eta_{-\infty} = \ln \alpha / \ln 2 \) and \( \eta_{\infty} = z \ln \alpha / \ln 2 \).

5 'Take away half’ RG transformation

Consider the positions of the orbit at \( \mu_\infty \) starting at \( x_0 = 0 \) up to time \( t = 2^N - 1 \) and from them extract the set of lengths \( l_j^{(0)}, j = 1, \ldots, 2^N \), that join closest adjacent positions. Each position is used only once so that the set \( l_j^{(0)} \) covers the attractor but not all the interval \( -1 \leq x \leq 1 \). To each length \( l_j^{(0)} \) assign the equiprobability \( p_j^{(0)} = 2^{-N} \). Following the standard procedure [1] we define the partition function

\[
Z_N^{(0)} (\beta, \tau) = \sum_j [p_j^{(0)}]^{\beta l_j^{(0)}} \tau ,
\]

and determine the (finite size) generalized dimensions \( D_{\beta,N} \equiv \tau / (\beta - 1) \) by requiring \( Z_N^{(0)} (\beta, \tau) = 1 \), \( (D_\beta = \lim_{N \to \infty} D_{\beta,N}) \). In Fig. 2 we show results for \( D_{\beta,N} \) for several values of \( z \).

We introduce now repeated length scale and restoring probability transformations

\[
l_j^{(k)} = 2^{-k \eta_j^{(0)}} \quad \text{and} \quad p_j^{(k)} = 2^{-k} p_j^{(0)} , \quad k = 1, \ldots, 2^N ,
\]

on the partition function \( Z_N^{(0)} (\beta, \tau) \). That is, to transform \( Z_N^{(0)} (\beta, \tau) \) into \( Z_N^{(1)} (\beta, \tau) \) replace \( l_j^{(0)} \) and \( p_j^{(0)} \) in Eq. (8) by \( l_j^{(1)} = 2^{-\eta_j^{(0)}} \) and \( p_j^{(1)} = 2^{-1} p_j^{(0)} \), re-
Fig. 2. The generalized dimensions $D_{\beta,N}$ for $z = 1.25$ ($\alpha(1.25) = 4.8323$, $N = 13$), $z = 1.5$ ($\alpha(1.5) = 3.4479$, $N = 16$) and $z = 2$ ($\alpha(2) = 2.5029$, $N = 14$).

Fig. 3. The length rescaling index $\eta$ in Eq. (10) for $z = 1.5$ ($\alpha(1.5) = 3.4479$, $N = 16$).

spectively, and proceed similarly any number $k$ of times. In the limit $N \to \infty$ the transformation eliminates one half of the multifractal points each time it is applied. By requiring $Z_N^{(k)}(\beta, \tau) = 1$ we obtain, again when $N \to \infty$, the following result for the rescaling exponent $\eta$ of the interval lengths

$$\eta = \frac{\beta}{\tau} = \frac{\beta}{\beta - 1} \frac{1}{D_{\beta}}. \quad (10)$$

As shown in Fig. 3, $\eta$ approaches two limiting values, $\eta_{-\infty} = D_{-\infty}^{-1}$ and $\eta_{\infty} = D_{\infty}^{-1}$ as $\beta \to -\infty$ and $\beta \to \infty$, respectively, and exhibits a singularity at $\beta = 1$. Except for values of $\beta$ in the vicinity of $\beta = 1$ the exponent $\eta$ is always close to either $\eta_{-\infty}$ or $\eta_{\infty}$ which correspond to the scaling factors $l_j^{(k)} = \alpha^{-k} l_j^{(0)}$ and $l_j^{(k)} = \alpha^{-zk} l_j^{(0)}$ described in the previous section. As also shown in Fig. 3, there
appears a gap in the permissible values of $\eta$,  
\[ \Delta \eta \equiv \eta_\infty - \eta_{-\infty} = \frac{1}{D_\infty} - \frac{1}{D_{-\infty}}, \]  
(11)

that is noticeably identical to the dynamical result Eq. (7), i.e. $\lambda_q = (1-q)^{-1} = \Delta \eta$.

Repeated application of the RG transformation when the limit $N \to \infty$ is taken before $k \to \infty$ leaves the generalized dimensions invariant, i.e. $D_\beta = D_\beta^{(k)}$ for all $k$. However when both $k, N \to \infty$ with $k = N$ the RG transformation leads to a fixed point $D_\beta^{(k)} \to D_\beta^{(\infty)}$ that has the form of a step function, $D_\beta^{(\infty)} = \ln 2/2 \ln \alpha$ for $\beta < 1$ and $D_\beta^{(\infty)} = \ln 2/2z \ln \alpha$ for $\beta > 1$, with a divergence at $\beta = 1$. See Fig. 4. [8].

### 6 Summary and discussion

Assisted by the known properties of the fixed-point map $g(x)$ we derived the expression for the sensitivity to initial conditions $\xi_t$ for the $z$-logistic map at the edge of chaos. As a consequence we corroborated the known expression for the entropic index $q$ in terms of $\alpha(z)$ [14] and obtained in addition the closely related expression for the $q$-Lyapunov coefficient $\lambda_q$ (known previously only for the particular case $z = 2$ [9] [10]). Our results are rigorous. The expression obtained is precisely that first derived, heuristically, by Lyra and Tsallis [14], Eq. (3), that relates $q$ to the smallest $D_\infty$ and largest $D_{-\infty}$ generalized dimensions of a multifractal set. In order to probe this connection
between dynamic and geometric properties of the multifractal attractor we introduced an RG approach that consists of sequentially removing remaining halves $2^{-k}$, $k = 1, 2, \ldots$ of points of the multifractal set while keeping the value of its partition function $Z_N^{(k)}(\beta, \tau)$ fixed. We determined the length rescaling index $\eta$ as a function of $\beta$ and observed that there is a gap $\Delta \eta$ of permissible values in this quantity equal to $\lambda_q = 1/(1-q)$, as the edges of the gap correspond to $\eta_{-\infty} = D_{-\infty}^{-1}$ and $\eta_{\infty} = D_{\infty}^{-1}$. This result can perhaps be better appreciated if we note that the orbit positions $|x_t|$ evolve according to two basic alternating movements observable at time intervals of the form $2^n$, one drives the iterate towards $|x| = 1$ as $\alpha^{-zn} = 2^{-\eta_{\infty}n}$ and the other one towards $x = 0$ as $\alpha^{-n} = 2^{-\eta_{-\infty}n}$. See the main diagonal subsequence positions in Figs. 1a and 1b. The net growth in the distance between two orbits is the ratio of these two power laws, that is $|\Delta x_t| = |\Delta x_0| 2^{\Delta \eta n}$ (see Section 3). Thus, Eq. (3) is obtained as the result of orbits that successively separate and converge, respectively, in relation to $D_{-\infty}$ and $D_{\infty}$. The power law for $\xi_t$ can only persist if $D_{-\infty} \neq D_{\infty}$.

Acknowledgments. We were partially supported by CONACyT grant P40530-F (Mexican agency).

References


[13] Here we refer to a simple form of rate of entropy increment that differs from the usual definition of the KS entropy (see Ref. [10]).
