Majority-vote on directed Barabási-Albert networks

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Abstract: On directed Barabási-Albert networks with two and seven neighbours selected by each added site, the Ising model was seen not to show a spontaneous magnetisation. Instead, the decay time for flipping of the magnetisation followed an Arrhenius law for Metropolis and Glauber algorithms, but for Wolff cluster flipping the magnetisation decayed exponentially with time. On these networks the Majority-vote model with noise is now studied through Monte Carlo simulations. However, in this model, the order-disorder phase transition of the order parameter is well defined in this system. We calculate the value of the critical noise parameter $q_c$ for several values of connectivity $z$ of the directed Barabási-Albert network. The critical exponents $\beta/\nu$, $\gamma/\nu$ and $1/\nu$ were calculated for several values of $z$.

Keywords: Monte Carlo simulation, vote, networks, nonequilibrium.

Introduction

It has been argued that nonequilibrium stochastic spin systems on regular square lattice with up-down symmetry fall in the universality class of the equilibrium Ising model [1]. This conjecture was found in several models that do not obey detailed balance [2, 3, 4]. Campos et al. [5] investigated the majority-vote model on small-world network by rewiring the two dimensional square lattice. These small-world networks, aside from presenting quenched disorder, also posses long-range interactions. They found that the critical exponents $\gamma/\nu$ and $\beta/\nu$ are different from the Ising model and depend on the rewiring probability. However, it was not evident that the exponent change was due to the disordered nature of the network or due to the presence of long-range interactions. Lima et al. [6] studied the majority-vote model on Voronoi-Delaunay random lattices with periodic boundary conditions. These lattices posses natural quenched disorder in their conecctions. They showed that presence of quenched connectivity disorder is enough to alter the exponents $\beta/\nu$ and $\gamma/\nu$ from the pure model and therefore that is a relevant term to such non-equilibrium phase-transition. Sumour and Shabat [7, 8]
investigated Ising models on directed Barabási-Albert networks [9] with the usual Glauber dynamics. No spontaneous magnetisation was found, in contrast to the case of undirected Barabási-Albert networks [10, 11, 12] where a spontaneous magnetisation was found lower a critical temperature which increases logarithmically with system size. More recently, Lima and Stauffer [13] simulated directed square, cubic and hypercubic lattices in two to five dimensions with heat bath dynamics in order to separate the network effects from the effects of directedness. They also compared different spin flip algorithms, including cluster flips [14], for Ising-Barabási-Albert networks. They found a freezing-in of the magnetisation similar to [7, 8], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetisation (in the usual sense) is consistent with the fact that if on a directed lattice a spin \( S_j \) influences spin \( S_i \), then spin \( S_i \) in turn does not influence \( S_j \), and there may be no well-defined total energy. Thus, they show that for the same scale-free networks, different algorithms give different results. Now we study the Majority-vote model on directed Barabási-Albert network and different from the Ising model, the order-disorder phase transition of order parameter well it is defined in this system. We calculate the \( \beta/\nu, \gamma/\nu, \) and \( 1/\nu \) exponents and these are different from the Ising model and depend on the values of connectivity \( z \) of the directed Barabási-Albert network.

**Model and Simulation**

We consider the majority-vote model, on directed Barabási-Albert Networks, defined [15, 16, 17] by a set of "voters" or spins variables \( \sigma \) taking the values \(+1\) or \( -1 \), situated on every site of a directed Barabási-Albert Networks with \( N \) sites, and evolving in time by single spin-flip like dynamics with a probability \( w_i \) given by

\[
w_i(\sigma) = \frac{1}{2} \left[ 1 - (1 - 2q)\sigma_i S \left( \sum_{\delta=1}^{k_i} \sigma_{i+\delta} \right) \right],
\]

where \( S(x) \) is the sign \( \pm 1 \) of \( x \) if \( x \neq 0 \), \( S(x) = 0 \) if \( x = 0 \), and the sum runs over all nearest neighbors of \( \sigma_i \). In this network, each new site added to the network selects \( z \) already existing sites as neighbours influencing it; the newly added spin does not influence these neighbours. The control parameter
q plays the role of the temperature in equilibrium systems and measures the probability of aligning antiparallel to the majority of neighbors.

To study the critical behavior of the model we define the variable $m = \sum_{i=1}^{N} \sigma_i / N$. In particular, we were interested in the magnetisation, susceptibility and the reduced fourth-order cumulant:

$$M(q) = \langle |m| \rangle_{av},$$

$$\chi(q) = N[\langle m^2 \rangle - \langle |m| \rangle^2]_{av},$$

$$U(q) = \left[ 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2} \right]_{av},$$

where $\langle \ldots \rangle$ stands for a thermodynamics average and $[\ldots]_{av}$, square brackets for averages over the 20 realizations.

These quantities are functions of the noise parameter $q$ and obey the finite-size scaling relations

$$M = N^{-\beta/\nu} f_m(x)[1 + \ldots],$$

$$\chi = N^{\gamma/\nu} f_\chi(x)[1 + \ldots],$$

$$\frac{dU}{dq} = N^{1/\nu} f_U(x)[1 + \ldots],$$

where $\nu$, $\beta$, and $\gamma$ are the usual critical exponents, $f_i(x)$ are the finite size scaling functions with

$$x = (q - q_c) N^{1/\nu}$$

being the scaling variable, and the brackets $[1 + \ldots]$ indicate corrections-to-scaling terms. Therefore, from the size dependence of $M$ and $\chi$ we obtained the exponents $\beta/\nu$ and $\gamma/\nu$, respectively. The maximum value of susceptibility also scales as $N^{\gamma/\nu}$. Moreover, the value of $q$ for which $\chi$ has a maximum, $q_{\chi}^{\text{max}} = q_c(N)$, is expected to scale with the system size as

$$q_c(N) = q_c + bN^{-1/\nu},$$

were the constant $b$ is close to unity. Therefore, the relations (7) and (9) are used to determine the exponents $1/\nu$. We have checked also if the calculated exponents satisfy the hyperscaling hypothesis

$$2\beta/\nu + \gamma/\nu = D_{\text{eff}}$$

$$3$$
in order to get the effective dimensionality, $D_{eff}$, for various values of $z$.

We have performed Monte Carlo simulation on directed Barabási-Albert networks with various values of connectivity $z$. For a given $z$, we used systems of size $N = 1000, 2000, 4000, 8000, \text{ and } 16000$. We waited 10000 Monte Carlo steps (MCS) to make the system reach the steady state, and the time averages were estimated from the next 10000 MCS. In our simulations, one MCS is accomplished after all the $N$ spins are updated. For all sets of parameters, we have generated 20 distinct networks, and have simulated 20 independent runs for each distinct network.

**Results and Discussion**

In Fig. 1 we show the dependence of the magnetisation $M$ and the susceptibility $\chi$ on the noise parameter, obtained from simulations on directed Barabási-Albert network with 16000 sites and several values of connectivity $z$. In the part (a) each curve for $M$, for a given value of $N$ and $z$, suggests that there is a phase transition from an ordered state to a disordered state. The phase transition occurs at a value of the critical noise parameter $q_c$, which is an increasing function the connectivity $z$ of the directed Barabási-Albert network. In the part (b) we show the corresponding behavior of the susceptibility $\chi$, the value of $q$ where $\chi$ has a maximum is here identified as $q_c$. In Fig. 2 we plot the Binder’s fourth-order cumulant for different values of $N$ and two different values of $z$. The critical noise parameter $q_c$, for a given value of $z$, is estimated as the point where the curves for different system sizes $N$ intercept each other. In Fig. 3, the phase diagram is shown as the dependence of the critical noise parameter $q_c$ on connectivity $z$ obtained from the data of Fig. 2.

The phase diagram of the majority-vote model on directed Barabási-Albert network shows that for a given network (fixed $z$) the system becomes ordered for $q < q_c$, whereas it has zero magnetisation for $q \geq q_c$. We notice that the increase of $q_c$ as a function of $z$ is slower than the one than in [17]. In the Fig. 4 we plot the dependence of the magnetisation at $q = q_c$ with the system size. The slopes of curves correspond to the exponent ratio $\beta/\nu$ of according to Eq. (5). The results show that the exponent ratio $\beta/\nu$ decreases when $z$ increases, see Table I.

In Fig. 5 we display the scalings for susceptibility at $\chi(q_c(N))$ (circle) for its maximum amplitude $\chi^{max}_N$, and $\chi(q_c(N))$ (square) obtained from the Binder’s cumulant versus $N$ for connectivity $z = 8$. The exponents ratios $\gamma/\nu$ are obtained from the slopes of the straight lines. For almost all the
values of $z$, the exponents $\gamma/\nu$ of the two estimates disagree (Table I). An increased $z$ means a tendency to increase the exponent ratio $\gamma/\nu$, see Table I, so that they disagree with the results of Luiz et al [17], where the values of exponents ratio $\gamma/\nu$ are almost all equal and with a slight tendency to decrease. Therefore we cannot use the Eq.(9), for fixed $z$, obtain the critical exponent $1/\nu$. In the Fig. 6 we show the critical behavior of $\beta/\nu$ and $\gamma/\nu$ as a function of connectivity $z$.

To obtain the critical exponent $1/\nu$, we calculated numerically $U'(q) = dU(q)/dq$ at the critical point for each values of $N$ at connectivity fixed $z$. The results are bad agreement with the scaling relation (7). Then, also we cannot calculate the exponents $1/\nu$, through this relation. Therefore we do not obtain to get the values of the exponents $1/\nu$ for each connectivity $z$.

The Table I resumes the values of $q_c$, the exponents $\beta/\nu$, $\gamma/\nu$, and the effective dimensionality of systems. For all values of $z$ the value $D_{eff} = 1$, which has been obtained from the Eq. (9), therefore when $z$ increases, $\beta/\nu$ decreases and $\gamma/\nu$ increases, thus providing the value of $D_{eff} = 1$ (along with errors). Therefore, the directed Barabási-Albert network has the same effective dimensionality as Erdős-Rényi’s random graphs [17]. J. M. Oliveira [15] showed which majority-vote model defined on regular lattice has critical exponents that fall into the same class of universality as the corresponding equilibrium Ising model. Campos et al [5] investigated the critical behavior of the majority-vote on small-world networks by rewiring the two-dimensional square lattice, Luiz et al [17] studied this model on Erdős-Rényi’s random graphs, and Lima et al [6] also studied this model on Voronoi-Delaunay lattice. The results obtained by these authors show that the critical exponents of majority-vote model belong to different universality classes.

Finally, we remark that our MC results obtained on directed Barabási-Albert network and undirected (in preparation) majority-vote model show that critical exponents are different from the results of [15] for regular lattice and of Luiz et al [17] for Erdős-Rényi’s random graphs.

**Conclusion**

In conclusion, we have presented a very simple nonequilibrium model on directed Barabási-Albert network [7, 8]. Different from the Ising model, in these networks, the Majority-vote model presents a second-order phase transition which occurs in model with connectivity $z > 1$. The exponents obtained are different from the other models. Nevertheless, our Monte Carlo simulations have demonstrated that the effective dimensionality $D_{eff}$ equals
Table 1: The critical noise $q_c$, the critical exponents, and the effective dimensionality $D_{\text{eff}}$, for directed Barabási-Albert network with connectivity $z$.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$q_c$</th>
<th>$\beta/\nu$</th>
<th>$\gamma/\nu q_c$</th>
<th>$\gamma/\nu q_c^{(N)}$</th>
<th>$D_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.434(3)</td>
<td>0.477(2)</td>
<td>0.064(8)</td>
<td>0.895(10)</td>
<td>1.018(9)</td>
</tr>
<tr>
<td>3</td>
<td>0.431(4)</td>
<td>0.444(1)</td>
<td>0.111(2)</td>
<td>0.904(12)</td>
<td>0.999(2)</td>
</tr>
<tr>
<td>4</td>
<td>0.431(3)</td>
<td>0.447(1)</td>
<td>0.104(2)</td>
<td>0.888(9)</td>
<td>0.998(3)</td>
</tr>
<tr>
<td>6</td>
<td>0.438(2)</td>
<td>0.435(2)</td>
<td>0.134(5)</td>
<td>0.861(3)</td>
<td>1.008(6)</td>
</tr>
<tr>
<td>8</td>
<td>0.444(5)</td>
<td>0.431(1)</td>
<td>0.138(2)</td>
<td>0.851(5)</td>
<td>1.000(2)</td>
</tr>
<tr>
<td>10</td>
<td>0.446(3)</td>
<td>0.421(2)</td>
<td>0.158(3)</td>
<td>0.834(7)</td>
<td>1.000(5)</td>
</tr>
<tr>
<td>20</td>
<td>0.458(4)</td>
<td>0.412(1)</td>
<td>0.178(2)</td>
<td>0.795(11)</td>
<td>1.002(2)</td>
</tr>
<tr>
<td>50</td>
<td>0.467(2)</td>
<td>0.375(4)</td>
<td>0.249(7)</td>
<td>0.735(17)</td>
<td>0.999(11)</td>
</tr>
<tr>
<td>100</td>
<td>0.474(3)</td>
<td>0.363(4)</td>
<td>0.269(5)</td>
<td>0.674(23)</td>
<td>0.999(9)</td>
</tr>
</tbody>
</table>

units, for all values of $z$, that are agree with the results de Luiz et al. However, when $z$ grows, the exponents at the critical point $q_c, \beta/\nu$ obtained by Binder’s cumulant decrease and the exponents $\gamma/\nu$ grow, satisfying the hyperscaling relation with $D_{\text{eff}} = 1$.

F.W.S. Lima has the pleasure to thank D. Stauffer for many suggestions and fruitful discussions during the development this work and also for the revision of this paper. I also acknowledge the Brazilian agency FAPEPI (Teresina-Piauí-Brasil) for its financial support.

References


Figure 1: Magnetisation and susceptibility as a function of the noise parameter $q$, for $N = 16000$ sites. From left to right, $z = 2, 3, 4, 6, 8, 10, 20, 50,$ and $100$. 
Figure 2: Binder's fourt-order cumulant as a function of $q$. In part (a) we have $z = 3$ and part (b) $z = 50$. 
Figure 3: The phase diagram, showing the dependence of critical noise parameter $q_c$ on connectivity $z$. 
Figure 4: \( \ln M(q_c) \) versus \( \ln N \). From bottom to top, \( z = 2, 4, 6, 10, 20, 50, \) and 100.
Figure 5: Plot of $\ln \chi_{\text{max}}(N)$ (circle) and $\ln \chi(q_c)$ (square) versus $\ln N$ for connectivity $z = 8.$
Figure 6: Critical behavior the $\beta/\nu$ and $\gamma/\nu$ exponents as a function of connectivity $z$. 