14.4–6: Gradient Vectors and Chain Rules

Calculus III

College of the Atlantic

1. Suppose $g(4) = 10$, and $g'(4) = 2$. Estimate $g(4.3)$.
2. Suppose $g(4, 6) = 10$, $g_x(4) = 2$, and $g_y(6) = 3$
   (a) Write down the differential of $g(x, y)$ at $(4, 6)$.
   (b) Estimate $g(4.3, 5.8)$.
3. Consider the function $T(x, y, z) = e^{-(x^2+y^2+z^2)}$.
   (a) Determine the gradient vector for general x, y, z.
   (b) Determine the gradient vector at the following points:
      i. $(0, 0, 0)$
      ii. $(1, 1, 1)$
      iii. $(1, 0, 0)$
   (c) What is the directional derivative of T in the $-\hat{z}$ direction at $(1, 0, 0)$?
   (d) What is the overall "shape" of the gradient vector field? How is this consistent with the level surfaces for this function?
4. For some unknown reason, a square room is slowly expanding. All of its walls are increasing at a rate of 0.2 meters/day. How fast is the area of the room increasing when the side of the room is 8 meters long?
5. Let $f(a, b) = a^2b^3$. At a particular moment in time, $a = 3$ and $b = 4$. At this moment, a is increasing at a rate of 2 units per second, while b is decreasing at 3 units per second. How fast is the function changing at this moment?
6. Suppose that z is a function of x and y: $z = f(x, y)$. And suppose that x and y are both functions of u and v: $x = g(u, v)$ and $y = h(u, v)$. How does z vary with u? To answer this question you will need to derive a new chain rule formula.

7. Let the temperature along a metal rod be given by $H(x,t)$, where: H is measured in Celsius degrees; x, the distance from the left end of the rod, is measured in centimeters; and t in minutes. Interpret the following equations:

   (a) $H(50,3) = 123$.
   (b) $H_t(50,3) = -2$.
   (c) $H_x(50,3) = -0.2$.
   (d) $H_{tx}(50,3) = 0.05$.

8. Let the temperature in a metal rod be given by the function $T(x,t) = 100e^{-t}\sin(\pi x)$, where t is measured in minutes and x in meters. The rod is one meter long. (So $0 \leq x \leq 1$.)

   (a) Sketch $T(x,0)$ and $T(x,0.1)$.

   (b) Using the two sketches you just drew, determine the signs of f_x, f_t, f_{xx}, and f_{xt} at $x = 0.2$.

   (c) Using the two sketches you just drew, determine the signs of f_x, f_t, f_{xx}, and f_{xt} at $x = 0.5$.

   (d) Using the two sketches you just drew, determine the signs of f_x, f_t, f_{xx}, and f_{xt} at $x = 0.8$.