The Seasonally Forced SEIR Model

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Below is a seasonally forced SEIR model with demographics.

$$\frac{dS}{dt} = \mu - \beta(t)SI - \mu S \,, \tag{1}$$

$$\frac{dE}{dt} = \beta(t)SI - \sigma E - \mu E \,, \tag{2}$$

$$\frac{dI}{dt} = \sigma E - \gamma I - \mu I \,, \tag{3}$$

$$\frac{dR}{dt} = \gamma I - \mu R \,. \tag{4}$$

- The Greek letters all represent model parameters: various rates.
- The capital Latin letters (SEIR) are the variables. These are the things we're trying to solve for. I.e., we want S as a function of time, I as a function of time, and so on.
- The quantity $\beta(t)$ is now a time-dependent parameter. This allows to have an infection rate that varies seasonally.

We will use the following for our time-dependent infection rate:

$$\beta(t) = \beta_0 \left(1 + \beta_1 \cos\left(\frac{2\pi t}{365} \right) \right) \,. \tag{5}$$

Let's explore and ponder.

1. I will send you a new copy of some basic SEIR code. Use the following parameters (which should already be in the code):

   - $\beta = 1000/365$.
   - People are exposed for an average of 8 days.
   - People are infectious for an average of 5 days.
   - People live an average of 50 years.

   I chose these values because they are what was used in Ottar N. Bjørnstad *Epidemics: models and data using R*. Springer Nature, 2022. These model parameters are roughly appropriate for measles.

2. We're going to be interested in endemicity and periodic small-ish outbreaks. So let's use different starting values than usual:

   - $S_0 = 0.06$
   - $E_0 = 0$
   - $I_0 = 0.001$
   - $R_0 = 1 - S_0 - E_0 - I_0$

These parameter values are fairly close to the endemic equilibrium values.

3. Before worrying about $\beta(t)$, run the model with the above parameters and initial conditions. Make a plot just of $I(t)$. You should see oscillations as the number of infecteds approach the equilibrium endemic value. By zooming in on the graph after the simulation has run for a while, estimate the period of the oscillations.

4. Spend some time pondering Eq. (5). Some questions to answer:

   (a) What does $\beta(t)$ look like?

   (b) What is the meaning of β_0 and β_1?

   (c) Why is there a $2\pi/365$ inside the cos?

   (d) Why is there at $+1$ in front of the cos?

   (e) Why is it cos and not sin?

   I would suggest using wolframalpha.com[1] to make some quick plots of $\beta(t)$.

5. Now that you have a deep understanding of $\beta(t)$, code it up! As always, if you have questions or hit roadblocks, let us know. For all of what follows, keep $\beta_0 = 1000/365$. Now let's try it out.

6. What happens if $\beta_1 = 0.05$? The plot may look a little weird. Focus on the longer-term behavior once $I(t)$ seems to settle in to a steady state. What is the period of oscillations. Does this make sense?

7. Hey. We should check to see if we can recover our unforced SEIR results. Set $beta_1 = 0$ and run the simulation. Your results should be the same as what you saw for Question 3.

8. Now let's make the seasonality stronger. Try $\beta_1 = 0.2$. Huh. What is the period of the $I(t)$ oscillations once it appears to settle into a steady state. Does this make sense?

[1] or desmos, if you're one of those people. ☺.