Linear Algebra

Exercises for Lecture Thirty: Image Compression and Change of Basis

Due Friday, November 22, 2013

Let T be a linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$. Let v_1, v_2 be the elements of the standard basis. (I.e., $v_1 = (1, 0)$ and $v_2 = (0, 1)$. Let the action of T on the basis vectors be given by:

$$T(v_1) = 3v_1 + v_2 \ , \tag{1}$$

$$T(v_2) = v_1 + 3v_2 \ . \tag{2}$$

1. What is $T(2v_1 + 4v_2)$? Answer this question without forming a matrix; use the linearity properties directly.
2. Determine the matrix representation A for the linear transformation T using the standard basis.
3. Calculate Ax, where $x = (2, 4)$. How does this compare to your answer to question 1?
4. Determine the eigenvalues and eigenvectors for A.
5. Normalize the eigenvectors. Call these eigenvectors u_1 and u_2.
6. Write v_1 as a linear combination of u_1 and u_2.
7. Write v_2 as a linear combination of u_1 and u_2.
8. Write down the matrix M that converts a vector in the eigenbasis into a vector in the standard basis. That is, let $c = (c_1, c_2)$ be the coefficients in the u basis:

   $$x = c_1u_1 + c_2u_2 \ . \tag{3}$$

   The matrix M takes the vector in the u basis and converts it to the v basis: $Mc = x$.
9. Find the inverse of M.
10. Evaluate $M^{-1}v_1$. What does this represent?
11. Determine $T(u_1)$ and $T(u_2)$.
12. Use your answer to the above question to determine a matrix representation for T in the u basis. Call this matrix B.
13. Verify that $B = M^{-1}AM$.