Lab One: Linear Systems Applications

Linear Algebra

College of the Atlantic

Due Friday, April 12, 2024

Instructions: Complete the following exercises in groups of two students; submit a single report for your group. If you want, you can write directly on the handout. Be sure to include complete explanations of the work you have done and justification for your conclusions. Scan your work and submit on google classroom. Please make sure that both lab partners' names are on the assignment.

Use `https://cocalc.com` for sage. Do not do these problems by hand.

These problems are exercises from Understanding Linear Algebra, by David Austin.

Do problem 1, and then do at least 2 of problems 2–5.

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1. Consider the system of linear equations:

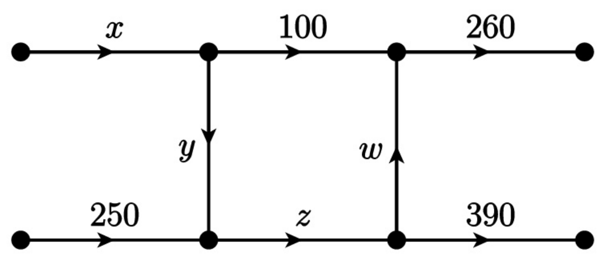
$$\begin{aligned} x_1 - 2x_2 + 4x_3 + x_4 - 4x_5 &= 3 \\ 2x_1 + x_2 + 3x_3 + x_4 + 3x_5 &= 1 \\ -4x_1 + 3x_2 - 11x_3 + x_4 + 1x_5 &= -7 \end{aligned}$$

   Represent this system as an augmented matrix, use Sage to find the RREF of the matrix, and fully describe the solution space of the system. If there are infinitely many solutions, describe them algebraically using parametric form.

2. Shown below is the traffic pattern in one part of the downtown area of a large city. The numbers and variables give the number of cars per hour traveling along each road segment.

   Any car that drives into an intersection must also leave the intersection. This means that the number of cars entering an intersection in an hour is equal to the number of cars leaving the intersection.

   Focus on the four "nodes" of the square, and think about equations that will have to hold based on the given information.



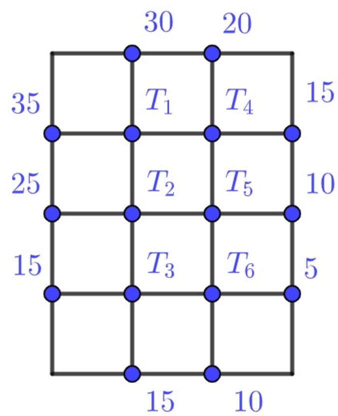
   (a) For the top left corner of the square, how many cars enter the intersection each hour? How many leave each hour? What equation results?

   (b) Reason similarly to you work in (a) to write a system of equations for the quantities x, y, z, and w; represent the system with an augmented matrix; find RREF; and and describe the set of solutions accordingly.

3. A typical problem in thermodynamics is to determine the temperature distribution at different points in a thin plate if you know the temperature around the boundary. For example, the thin plate might represent a cross section of a metal beam with negligible heat flow in the direction perpendicular to the plate.

   For this example, let $T_1, \ldots, T_6$ be the respective temperatures at the six pictured nodes inside the beam in the figure below. The temperature at a node is approximately the average of the four nearest nodes; in this exercise, we will assume that this average tells us the exact temperature at a given node. For instance,

$$T_1 = \frac{30 + 35 + T_2 + T_4}{4} \quad \text{or} \quad 4T_1 - T_2 - T_4 = 65.$$



   In more sophisticated[1] applications, the approximation becomes better the closer the points are together or as we add more and more into the grid.

   In the space to the right of the figure, write a system of equations for the quantities $T_1, \ldots, T_6$. Then, in the space below, represent the system with an augmented matrix; find RREF; and and describe the set of solutions accordingly. Write at least one sentence to explain the meaning of your result(s).

   Helpful Sage hint: If you have a matrix `B` containing rational entries (that is, fractions), you can obtain a decimal approximation using `B.numerical_approx(digits=4)`. You may, of course, change "4" to any other appropriate value.

[1]What is being described here is a *relaxation algorithm*, which is a numerical approach to solving the Laplace equation, which is an equation that arises in heat flow, electrostatics, and elsewhere.

4. The fuel inside model rocket motors is a black powder mixture that ideally consists of 60% charcoal, 30% potassium nitrate, and 10% sulfur by weight.

   Suppose you work at a company that makes model rocket motors. When you come into work one morning, you learn that yesterday's first shift made a perfect batch of fuel. The second shift, however, used a different formula that produced 50% charcoal, 20% potassium nitrate and 30% sulfur. Then the two batches were mixed together. A chemical analysis shows that there are **100.3** pounds of charcoal in the mixture and **46.9** pounds of potassium nitrate.

   (a) Let x be the number of pounds of fuel produced by the first shift, and y the number of pounds produced by the second shift. Set up a system of linear equations in x and y that reflects the given information in the problem statement.

   (b) How many pounds of fuel were produced by the first and second shifts? Why?

   (c) How many pounds of sulfur were in the combined mixture of fuel? Why?

5. This exercise is about balancing chemical reactions.

(a) Chemists denote a molecule of water as (H_2O,) which means it is composed of two atoms of hydrogen (H) and one atom of oxygen (O). The process by which hydrogen burns is described by the chemical reaction

$$x\,H_2 + y\,O_2 \to z\,H_2O \tag{1}$$

This means that x molecules of hydrogen (H_2) combine with y molecules of oxygen (O_2) to produce z water molecules. The number of hydrogen atoms is the same before and after the reaction; the same is true of the oxygen atoms.

i. In terms of x, y, and z how many hydrogen atoms are there before the reaction? How many hydrogen atoms are there after the reaction? Find a linear equation in x, y, and z by equating these quantities.

ii. Find a second linear equation x, y, and z by equating the number of oxygen atoms before and after the reaction.

iii. Find the solutions to this linear system. Why are there infinitely many solutions?

iv. In this chemical setting, x, y, and z should be positive integers. Find the solution where x, y, and z are the smallest possible positive integers.

(b) Now consider the reaction where potassium permanganate and manganese sulfate combine with water to produce manganese dioxide, potassium sulfate, and sulfuric acid:

$$x_1\,\mathrm{KMnO_4} + x_2\,\mathrm{MnSO_4} + x_3\,\mathrm{H_2O} \longrightarrow x_4\,\mathrm{MnO_2} + x_5\,\mathrm{K_2SO_4} + x_6\,\mathrm{H_2SO_4}. \tag{2}$$

As in the previous exercise, find the appropriate values for $x_1, x_2, \ldots, x_6$ to balance the reaction.