Chapter C13: Angular Momentum

C13.1: The Case of the Rotating Person

Consider the situation when someone spins, moves her or his arms in, and spins faster. It turns out that this process does not conserve rotational kinetic energy. Could there be something else that *is* conserved?

C13.2: Cross Product

The cross product is, like the dot product, a way to "multiply" two vectors together. Note that "·" and "×" do not mean the same thing! The dot product takes two vectors and turns them into a scalar. The cross product takes two vectors and returns another vector.

$$\text{mag}(\vec{u} \times \vec{w}) = uw \sin \theta \tag{1}$$

The direction of $\vec{u} \times \vec{w}$ is perpendicular to the plane that contains $\vec{u}$ and $\vec{w}$ and is given by the right hand rule.

Another way to think of the cross product is as follows:

$$\text{mag}(\vec{u} \times \vec{w}) = uw_\perp = u_\perp w , \tag{2}$$

where $u_\perp$ is read "the part of u perpendicular to v, or "u perp".
In components:

$$\vec{u} \times \vec{w} \equiv \begin{bmatrix} u_y w_z - u_z w_y \\ u_z w_x - u_x w_z \\ u_x w_y - u_y w_x \end{bmatrix} \tag{3}$$

We won't use this equation explicitly, but it is perhaps comforting to know that it exists. Note that $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

Examples

1. Let $\vec{u}$ be a displacement vector of 2 meters that points due east, and let $\vec{w}$ be a vector with a magnitude of 2 meters that points due south.

   (a) Find $\vec{u} \times \vec{w}$.

(b) Find $\vec{u} \cdot \vec{w}$.

2. Let $\vec{v}_1$ be a displacement vector of 2 meters that points due east, and let $\vec{v}_2$ be a vector with a magnitude of 2 meters that points 45 degrees north of west.

   (a) Find $\vec{v}_1 \times \vec{v}_2$.
   (b) Find $\vec{v}_1 \cdot \vec{v}_2$.

C13.3 The Angular Momentum of a Particle

- An object's angular momentum relative to the point O is defined as:

$$\vec{L} \equiv m(\vec{r} \times \vec{v}) \,, \tag{4}$$

  where $\vec{v}$ is the object's velocity and $\vec{r}$ is a vector from point O to the object.

- Note that the particle need not be moving in a circle for it to have angular momentum.

- An easier way to think of the above formula: $L = mr_\perp v$, where $r_\perp$ is the component of $\vec{r}$ perpendicular to $\vec{v}$.

- Yet another way to think of the above formula: $L = mrv_\perp$, where $v_\perp$ is the component of $\vec{v}$ perpendicular to $\vec{r}$.

- However, if the particle is moving in a circle, then

$$\vec{L} = mr^2\vec{\omega} \tag{5}$$

C13.4 The Angular Momentum of a Rigid Object

For a solid object of moment of inertia I rotating with an angular velocity of $\vec{\omega}$,

$$\vec{L} = I\vec{\omega} \,. \tag{6}$$

C13.5 The Angular Momentum of a Moving Object

If we have an object rotating about its axis and the object is also moving as a whole around a point O, then

$$\vec{L} = \vec{L}^{\text{cm}} + \vec{L}^{\text{rot}} \tag{7}$$
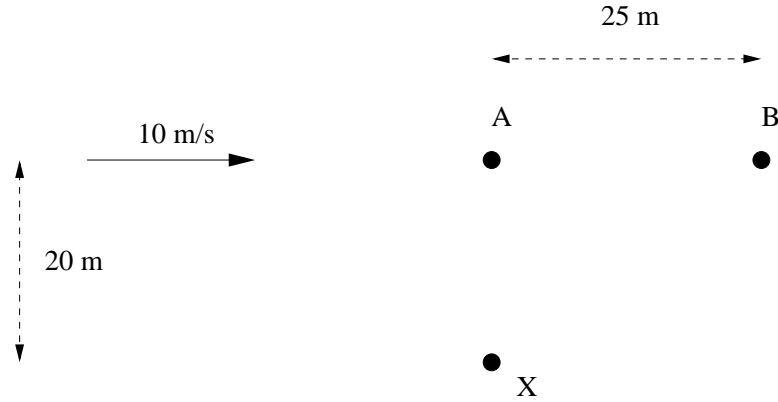
25 m

A B

10 m/s

20 m

X

Figure 1:

C13.6: Twirl and Torque

Recall that in chapter four we referred to a momentum transfer as an impulse $[d\vec{p}]$. And the rate of momentum transfer was referred to as force:

$$\vec{F} \equiv \frac{[d\vec{p}]}{dt} \ . \tag{8}$$

We call an angular momentum transfer *twirl*: $[d\vec{L}]$. And the rate of angular momentum transfer is *torque*:

$$\vec{\tau} \equiv \frac{[d\vec{L}]}{dt} \ . \tag{9}$$

Force is to linear momentum as torque is to angular momentum. A better name for torque is "turning force." We won't do much explicitly with torque, but it's good to know about this term.

Examples:

1. You are standing at point X in Fig. 1. A 2 kg bird flies by you. What is its angular momentum with respect to point X when the bird is at point A? What is its angular momentum at point B?

2. A 20 kg bowling ball rotates in space, making 4 revolutions every second. What is its angular momentum?