Chapter N2: Vector Calculus

Don't be intimidated by the title of this chapter. The main goal of this chapter is to learn how to describe motion. ("Describing motion" might have been a friendlier title.) We aren't interested in asking what causes motion—we talked about that in N1 and it will also be the topic of N3. Here, we develop a mathematical vocabulary that will let us describe the motion we observe.

N2.1 & N2.2: The Time-Derivative of a Vector and Velocity

In Book C, we often wrote

$$v = \frac{\Delta x}{\Delta t} \tag{1}$$

where dx was understood to be a change in position. (Sometimes we used dx instead of Δx.) How small must Δx and Δt be? We said that Δt should be small enough so that Δx doesn't change much during dt.

But we can do better than this. We define

$$v(t) = \frac{dx(t)}{dt} \equiv \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \, . \tag{2}$$

Here $x(t)$ means the value of x at time t. This equation is just "rise over run", where the "run" (Δt) is very very tiny.

The quantity $v(t)$ is the instantaneous velocity—the velocity at the instant in time t. Geometrically, $v(t)$ is the instantaneous slope of $x(t)$.

The derivative of a vector is defined as:

$$\frac{d\vec{r}(t)}{dt} \equiv \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{bmatrix} \tag{3}$$

N2.3: The Definition of Acceleration

Acceleration is defined as the rate of change of velocity:

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} \, . \tag{4}$$

Thus, the acceleration is derivative of the velocity

$$\vec{a} = \frac{d\vec{v}}{dt} \, . \tag{5}$$

The average acceleration $\vec{a}_{\Delta t}$ during a time interval Δt is defined as:

$$\vec{a}_{\Delta t} \equiv \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} \,. \tag{6}$$

Example: A car goes from zero to 60 miles per hour in 12 seconds. What is the magnitude of its average acceleration during these 12 seconds?

N2.4: Motion Diagrams

For both of the motion diagrams, assume that the time between dots is 0.1 seconds.

1. For the motion diagram in Fig. 1:
   (a) What is the average velocity between 1 and 2?
   (b) What is the average velocity between 2 and 3?
   (c) What is the approximate acceleration at 2?
2. For the motion diagram in Fig.2:
   (a) What is the magnitude of the average velocity between 1 and 2?
   (b) What is the magnitude of the average velocity between 2 and 3?
   (c) What is the magnitude of the acceleration at point 2?

N2.5: Numerical Results from Motion Diagrams

Remember that velocities and accelerations are vectors. So to add and subtract them, you can't just add and subtract their magnitudes.

N2.6: Uniform Circular Motion

The main equation:

$$a = \frac{v^2}{r} \,, \tag{7}$$

where the direction of $\vec{a}$ is directly toward the center of the circle. This formula only applies if the object is moving in a circle at a constant speed!

Figure 1: A motion diagram

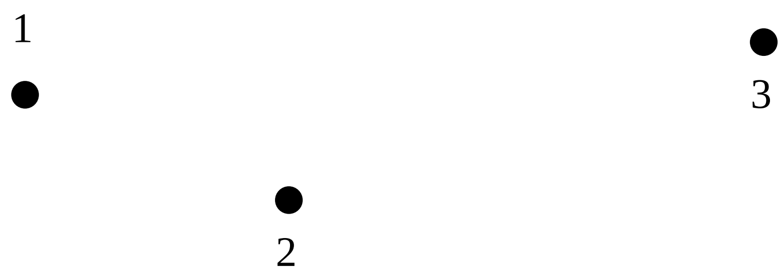

Figure 2: Another motion diagram

Practice

Draw velocity and acceleration arrows for the following two motion diagrams. Estimate the magnitude of the acceleration in each case.