Homework 04

Physics II

Due Friday, April 22, 2022

College of the Atlantic. Spring 2022

There are two parts to this assignment.

Part 1: WeBWorK. Do Homework 04 which you will find on your WeBWorK page. I recommend doing the WeBWorK part of the homework first. This will enable you to benefit from WeBWorK's instant, if not necessarily friendly, feedback before you do part two.

Part 2: Not WeBWorK. Below are some non-WeBWorK problems.

- If you want, you can do these problems in pairs and hand in only one write-up.
- "Hand in" the problem on google classroom. You can take a picture of your work, or type up your work, or scan your work.

Figure 1: Navstar-2F satellite of the Global Positioning System. Figure source, USAF, released into the public domain. `https://en.wikipedia.org/wiki/GPS_Block_IIF#/media/File:Navstar-2F.jpg`.

Background and overview

- The goal of this problem is to see how special relativity is needed in order for the Global Positioning System to work. You'll work through a very simplified example that, although not close to realistic, will nevertheless show how relativity is needed. It will also tie together a bunch of what we've learned in this and previous units.
- GPS satellites constantly broadcast a signal the specifies what time the signal is sent, according to an atomic clock on the satellite, and where the satellite is at when it sends the signal. The satellite are not in a stationary orbit. They are zipping around the earth at roughly 4000 m/s. So they are in a different reference frame than GPS users on the surface of the earth.

- A GPS receiver uses information sent from several different satellites to figure out where it is. As you know from navigation devices in smartphones and cars, a GPS receiver can figure out its location on earth to within a few meters.
- This is a modified version of problem R6R.2 from Tom Moore *Six Ideas that Shaped Physics: Unit R (second edition)*, (2003). Moore's problem is adapted from an example in chapter 4 of J.B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (2002).
- Below, I lead you through a multi-step problem. If anything isn't clear or you have any questions whatsoever, please don't hesitate to reach out.
- In the scenario we'll analyze below, we'll pretend that the earth is flat, and that we only have to think about signals from two satellites.

1. Suppose you simultaneously receive a signal from two satellites. The position of the satellites is x'_A = -3000 km and x'_B = 9000 km. Both satellites tell you that they sent the signal at $t' = 0$. The signals travel at the speed of light. Ignoring special relativity, and assuming that the signals travel at the same speed in your frame, what is your position?
2. Repeat the above problem but instead of coming up with a number for your answer, write your position x as a formula involving x'_A and x'_B. (You should find that $x = (1/2)(x'_B + x'_A)$.)
3. Now let's analyze the situation using relativity. We'll consider the following situation. Suppose that both satellites are moving at a constant velocity of $\beta = 3/5$ with respect to the flat earth. So the two satellites are both in the same inertial reference frame. We'll assume that the satellites are orbiting low enough that we can ignore their altitude. The two satellites emit a signal simultaneously at $t' = 0$ in their reference frame. Call the emissions of these signals events A and B. Event A is to your left, and B is to the right. The locations of events A and B in the satellite frame are $x'_A = 1$ s and $x'_B = 5$ s. Signals from A and B travel to the right and left at the speed of light, and meet somewhere in the middle. Call this meeting point event C.
   (a) Draw a two-observer spacetime diagram for this situation, labeling events A, B, and C.
   (b) What are the spacetime coordinates of event C in the satellite frame? Hint: What is happening here is basically the "radar method" in the satellite frame for finding the spacetime coordinates of an event.
   (c) Repeat the above problem, but instead of a numerical answer, write your answer for t'_C and x'_C in terms x'_A and x'_B.
4. Let's figure out x_c, the position where the signals cross in the earth reference frame. To do so, take your answers for t'_C and x'_C and use the Lorentz transformation. You should find that:
$$x_c = \gamma\left(\frac{\beta}{2}(x'_B - x'_A) + \frac{1}{2}(x'_A + x'_B)\right) . \tag{1}$$
5. Before we use the above result to think about GPS, a little bit of math. Remember the binomial approximation?
$$(1 + x)^a \approx 1 + ax . \tag{2}$$

Let's use the binomial approximation on γ. Note that $\gamma = (1-\beta^2)^{-1/2}$.

(a) What is the binomial approximation for γ?

(b) Suppose $\beta = 3.9$ km/s. Convert this speed to SR units.

(c) Use the binomial approximation to evaluate γ if $\beta = 3.9$ km/s. Write your answer in the form $\gamma = 1 + \alpha$, where α will be a number.

(d) Make a note to yourself that $\alpha \ll \beta$.

6. Take a look at Eq. (1). Note that $\frac{1}{2}(x'_A + x'_B)$ is just x'_C. Using this along with the binomial approximation for γ, show that Eq. (1) can be rewritten as:

$$x_c = (1+\alpha)\left(x'_C + \frac{\beta}{2}(x'_B - x'_A)\right) . \tag{3}$$

7. Multiply out this equation and you'll get four terms on the right. But in Problem 5d, you noted that $\alpha \ll \beta$. This means that we can ignore any term that has an α in it! So drop the α terms and rearrange and show that Eq. (3) can be written as:

$$x_c - x'_C = \frac{\beta}{2}(x'_B - x'_A) . \tag{4}$$

8. The above equation is what we are looking for. On the left-hand side is the difference between the position we would measure if we ignored relativity (x'_C) and the actual position of event C in the earth frame (x_C). Suppose $\beta = 3.9$ km/s, $x'_B = 9000$ km and $x'_A = -3000$ km. Solve for $x_C - x'_C$. You should get a number between 50 and 100 meters. This means that if we were ignorant of special relativity, GPS would not work.

The above example is obviously too simple to fully capture all of what is going on in GPS. In reality we would need to account for the fact that the earth is not flat and the earth is not stationary (it rotates). We would also need to account for general relativity, which says that massive objects like the earth bend spacetime itself. Despite all this, this example illustrates that special relativity is essential in order to use GPS to figure out location.

Some optional problems

1. Use the Lorentz transformation equations to show that the square of the spacetime interval is the same in all inertial reference frames. This problem involves a moderate amount of algebra but no calculus.

2. Consider two identical twin emperor penguins, each of which is going to live to be exactly 20 years old, as measured by their own biological clocks. Since infancy, one twin lives at the equator and the other lives at the south pole. How much longer will the equatorial twin live than the south-pole twin? In thinking about this problem, it will probably be easiest to use a reference frame based in the center of the earth.

Figure 2: Emperor Penguins. Image in the public domain. Image ID: corp2417, NOAA Corps Collection Photographer: Giuseppe Zibordi Credit: Michael Van Woert, NOAA NESDIS, ORA. `https://commons.wikimedia.org/wiki/File:Kaiserpinguine_mit_Jungen.jpg`.